MATH 147 SPRING 2021: SOLUTIONS TO EXAM 3

You must show all work to receive full credit. Each problem is worth 20 points.

1. Let C be the helix $\mathbf{r}(t) = (2\cos(t), 2\sin(t), 2t)$, with $0 \le t \le 4\pi$. Set up, but do not calculate, the integrals giving:

(i) $\int_C \mathbf{F} \cdot d\mathbf{r}$, for $\mathbf{F} = xyz\vec{i} + (xz + y^3)\vec{j} + z\vec{k}$.
(ii) The arc length of C.
(iii) $\int_C e^{x^2+y^2+z^2}\ ds$.

Solution. $\mathbf{r}'(t) = (-2\sin(t), 2\cos(t), 2)$ and $||\mathbf{r}'(t)|| = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2 + 2^2} = \sqrt{8}$.

For (i) $\mathbf{F}(\mathbf{r}(t)) = (8t\cos(t)\sin(t), 4t\cos(t) + 8\sin^3(t), 2t)$, thus

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{4\pi} (8t\cos(t)\sin(t), 4t\cos(t) + 8\sin^3(t), 2t) \cdot (-2\sin(t), 2\cos(t), 2)\ dt \\ &= \int_0^{4\pi} -16t\cos(t)\sin^2(t) + 8t\cos^2(t) + 16\sin^3(t)\cos(t) + 4t\ dt.\end{aligned}$$

For (ii), arc length $= \int_0^{4\pi} ||\mathbf{r}'(t)||\ dt = \int_0^{4\pi} \sqrt{8}\ dt$.

For (iii) $\int_C e^{x^2+y^2+z^2}\ ds = \int_0^{4\pi} e^{4\cos^2(t)+4\sin^2(t)+4t^2}\ \sqrt{8}\ dt$.

2. Describe heuristically the formal **definition** of $\int\int_S f(x, y, z)\ dS$, the surface integral of the scalar function $f(x, y, z)$ over the surface S. Note, this is not asking you how to calculate $\int\int_S f(x, y, z)\ dS$.

Solution. Subdivide S into small subregions S_i having surface area ΔS. Choose a point $P_i \in S_i$. Evaluate $f(x, y, z)$ at P_i, to get $f(P_i)$. Form the Riemann sum $\sum_i f(P_i)\Delta S$. Then take the limit of the Riemann sums as ΔS tends to 0.

3. Let S be the closed cylinder, with radius 6 and height 5, whose base is in the xy-plane. Calculate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ with respect to the outward normal, for $\mathbf{F} = (y^2x^2 + x)\vec{i} + (x^2z^2 + y)\vec{j} + (x^2y^2 + z)\vec{k}$.

Solution. Let B denote the solid enclosed by S. By the Divergence Theorem, we have

$$\begin{aligned}\int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int\int_B \text{div } \mathbf{F}\ dV \\ &= \int\int\int_B 3\ dV \\ &= 3 \cdot \text{volume}(B) \\ &= 3\pi 6^2 \cdot 5 \\ &= 540\pi.\end{aligned}$$

4. Let B_0 denote the box $[0, a] \times [0, b] \times [0, c]$ in xyz-space and B the box obtained by translating the origin to the point (x_0, y_0, z_0). Here, we assume $a, b, c > 0$. Rewrite the triple integral $\int\int\int_B 2x^2y + 5xz\ dV$ in terms of u, v, w, so that the new domain of integration is the unit box $[0, 1] \times [0, 1] \times [0, 1]$. Do not calculate the resulting expression.

Solution. We take $G(u,v,w) = (au + x_0, bv + y_0, cw + z_0)$. Then

$$\text{Jac}(G) = \det \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc.$$

Note $|\text{Jac}(G))| = abc$. Then

$$\int\int\int_B 2x^2y + 5xz \; dV = \int_0^1 \int_0^1 \int_0^1 \{2(au + x_0)^2(bv + y_0) + 5(au + x_0)(cw + z_0)\} \cdot abc \; du \; dv \; dw.$$

5. Let S denote the solid triangle with vertices (1,0,0), (0,2,0), (0,0,4). Calculate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ with respect to the upward normal, for $\mathbf{F} = 8\vec{i} + 4\vec{j} + 2\vec{k}$.

Solution. S is part of the plane passing through the given points, so we find the normal vector to the plane. The vectors $-\vec{i} + 2\vec{j}$ and $-\vec{i} + 4\vec{k}$ form two sides of S, so that

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = 8\vec{i} + 4\vec{j} + 2\vec{k}$$

is normal to the surface, and points upward. This vector has length $\sqrt{84}$, so that the unit normal to S is $\mathbf{n} = \frac{8}{\sqrt{84}}\vec{i} + \frac{4}{\sqrt{84}}\vec{j} + \frac{2}{\sqrt{84}}\vec{k}$. Thus, on S, $\mathbf{F} \cdot \mathbf{n} = \frac{84}{\sqrt{84}} = \sqrt{84}$. Therefore,

$$\begin{aligned}
\int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int_S \mathbf{F} \cdot \mathbf{n} \; dS \\
&= \int\int_S \sqrt{84} \; dS \\
&= \sqrt{84} \cdot \text{surface area}(S) \\
&= \sqrt{84} \cdot \frac{\sqrt{84}}{2} \\
&= 42.
\end{aligned}$$

Note that the surface area of S is one half the area of the parallelepiped spanned by the $-\vec{i}+2\vec{j}$ and $-\vec{i}+4\vec{k}$, and the latter area is the length of the normal vector $8\vec{i} + 4\vec{j} + 2\vec{k}$, which is $\sqrt{84}$.

Alternately, the equation of the plane on which S lies is $4(x-1) + 2y + z = 0$. Using the parametrization $G(u,v) = (u, v, -4u - 2v + 4)$, with $0 \le u \le 1$ and $0 \le v \le -2u + 2$ defining D in the uv-plane, we have

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -4 \\ 0 & 1 & -2 \end{vmatrix} = 4\vec{i} + 2\vec{j} + \vec{k},$$

which is the upward normal vector. $\mathbf{F}(G(u,v)) = 8\vec{i} + 4\vec{j} + 2\vec{k}$, so that $\mathbf{F}(G(u,v)) \cdot \mathbf{T}_u \times \mathbf{T}_v = 42$. Thus,

$$\begin{aligned}
\int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int_D \mathbf{F}(G(u,v)) \cdot \mathbf{T}_u \times \mathbf{T}_v \; dA \\
&= \int_0^1 \int_0^{-2u+2} 42 \; dv \; du \\
&= 42 \cdot \text{area}(D) \\
&= 42.
\end{aligned}$$

Optional Bonus Problem. Let $f(x,y,z) = \frac{-1}{(x^2+y^2+z^2+1)}$ and L denote the (infinite) half line starting at (0,0,0) and passing through the point $P = (a,b,c)$. Show by direct calculation that the line integral $\int_L \nabla f \cdot d\mathbf{r}$ is independent of the point P. Here ∇f denotes the gradient of $f(x,y,z)$. (10 points)

Solution. $\nabla f = \frac{1}{(x^2+y^2+z^2+1)^2} \cdot (2x, 2y, 2z)$. L is given by $\mathbf{r}(t) = (at, bt, ct)$, with $0 \leq t < \infty$, so $\mathbf{r}'(t) = (a, b, c)$. Therefore, $\nabla f(\mathbf{r}(t)) = \frac{1}{(a^2t^2+b^2t^2+c^2t^2+1)^2} \cdot (2at, 2bt, 2ct)$. Therefore,

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \frac{2a^2t + 2b^2t + 2c^2t}{((a^2+b^2+c^2)t^2+1)^2} = \frac{2t(a^2+b^2+c^2)}{((a^2+b^2+c^2)t^2+1)^2}.$$

Therefore,

$$\begin{aligned}
\int_L \nabla f \cdot d\mathbf{r} &= \int_0^\infty \frac{2t(a^2+b^2+c^2)}{((a^2+b^2+c^2)t^2+1)^2}\ dt \\
&= \int_1^\infty \frac{1}{u^2}\ du, \text{ using } u\text{-substitution with } u = (a^2+b^2+c^2)t^2+1 \\
&= \lim_{p\to\infty} \left. \frac{-1}{u} \right|_1^p \\
&= 1.
\end{aligned}$$